

CENTRALISERS OF DEHN TWIST AUTOMORPHISMS OF FREE GROUPS

MORITZ RODENHAUSEN AND RICHARD D. WADE

ABSTRACT. We refine Cohen and Lustig's description of centralisers of Dehn twists of free groups. We show that the centraliser of a Dehn twist of a free group has a subgroup of finite index that has a finite classifying space. We describe an algorithm to find a presentation of the centraliser. We use this algorithm to give an explicit presentation for the centraliser of a Nielsen automorphism in $\text{Aut}(F_n)$.

1. INTRODUCTION

Given a group G and an element $g \in G$, a natural question is study the centraliser $C(g)$ of g in G . In several classes of groups, such as hyperbolic and $\text{CAT}(0)$ groups [4], and mapping class groups [9], centralisers of elements are reasonably well-understood. In $\text{Out}(F_n)$, Feighn and Handel classified abelian subgroups in $\text{Out}(F_n)$ by studying centralisers of elements [7] and the centraliser of a *fully irreducible* element is virtually-cyclic [1]. However, it is not clear how centralisers of elements behave in general.

A *Dehn twist* $D = (\mathcal{G}, (\gamma_e)_{e \in E(\Gamma)})$ is defined by a graph of groups \mathcal{G} along with an element γ_e in the centre of each edge group of \mathcal{G} . An isomorphism $\rho : \pi_1(\mathcal{G}, v) \rightarrow F_n$ then defines elements $D_{*v} \in \text{Aut}(F_n)$ and $\widehat{D} \in \text{Out}(F_n)$. The class of all such automorphisms includes Nielsen automorphisms and Whitehead automorphisms of infinite order.

It is illuminating to look first at the analogous elements in mapping class groups: comparable to \widehat{D} is a *multitwist* M given by taking disjoint, pairwise non-isotopic, simple closed curves c_1, \dots, c_m on a surface Σ , non-zero integers a_1, \dots, a_m , and defining $M = T_{c_1}^{a_1} \cdots T_{c_m}^{a_m}$, where T_{c_i} is the Dehn twist (in the classical sense) in the curve c_i . In the centraliser $C(M)$ of M in the mapping class group one has the finite index subgroup $C^0(M)$ consisting of homeomorphisms that fix each curve c_i (up to homotopy). When we cut along these curves we obtain a set of

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punctured surfaces $\Sigma_1, \dots, \Sigma_k$ and an exact sequence

$$1 \rightarrow \mathbb{Z}^m \rightarrow C^0(M) \rightarrow \bigoplus_{i=1}^k MCG(\Sigma_k) \rightarrow 1,$$

where the free abelian group in this sequence is generated by T_{c_1}, \dots, T_{c_m} and the right hand surjection is given by restriction to the Σ_i . This allows one to study $C(M)$ through mapping class groups of open sub-surfaces.

Our main theorem establishes a similar picture when we have an *efficient* Dehn twist $D = (\mathcal{G}, (\gamma_e)_{e \in E(\Gamma)})$ (Definition 4.8, below) and $\pi_1(\mathcal{G}, v) \cong F_n$. In this situation, each vertex group G_w of \mathcal{G} is a finitely generated free group and each edge group G_e is cyclic. For a vertex group G_w , the generating elements of the edge groups adjacent to w give a set of conjugacy classes \mathcal{C}_w in G_w . Rather than having mapping class groups of punctured subsurfaces, we instead have the relative automorphism groups $\text{Out}(G_w, \mathcal{C}_w)$ and $\text{Aut}(G_w, \mathcal{C}_w)$ that consist of (outer) automorphisms that fix each conjugacy class in \mathcal{C}_w .

There is an action of $C(\widehat{D})$ on the underlying graph Γ of \mathcal{G} , and we define $C^0(\widehat{D})$ to be the finite-index subgroup consisting of automorphisms that act trivially on Γ . In the $\text{Aut}(F_n)$ case, this action fixes a chosen basepoint v of Γ , so lies in the subgroup $\text{Aut}(\Gamma, v)$ of $\text{Aut}(\Gamma)$ consisting of graph isometries that fix v .

Theorem 5.10. *Let D be an efficient Dehn twist on a graph of groups \mathcal{G} with $\pi_1(\mathcal{G}, v) \cong F_n$. Let $C(\widehat{D})$ and $C(D_{*v})$ be the centralisers of D in $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ respectively. There exist homomorphisms:*

$$\begin{aligned} \bar{\alpha}: C(\widehat{D}) &\rightarrow \text{Aut}(\Gamma), \\ \bar{\beta}: C(D_{*v}) &\rightarrow \text{Aut}(\Gamma, v), \end{aligned}$$

*such that their kernels $C^0(\widehat{D})$ and $C^0(D_{*v})$ are finite index subgroups that fit into the exact sequences:*

$$\begin{aligned} 1 \rightarrow DO(\mathcal{G}) \rightarrow C^0(\widehat{D}) &\rightarrow \bigoplus_{w \in V(\Gamma)} \text{Out}(G_w, \mathcal{C}_w) \rightarrow 1, \\ 1 \rightarrow DA(\mathcal{G}) \rightarrow C^0(D_{*v}) &\rightarrow \text{Aut}(G_v, \mathcal{C}_v) \oplus \left(\bigoplus_{w \neq v} \text{Out}(G_w, \mathcal{C}_w) \right) \rightarrow 1, \end{aligned}$$

where $DA(\mathcal{G})$ and $DO(\mathcal{G})$ are free abelian groups of Dehn twists of rank equal to the number of geometric edges of \mathcal{G} .

Our main inputs are the theory of *automorphisms of graphs of groups* (summarised in Section 2) and a theorem of Cohen and Lustig [5] showing that an element of the centraliser of an efficient Dehn twist may be

represented by an automorphism of \mathcal{G} . It follows that there is a group $\text{Aut}(\mathcal{G}, \widehat{D})$ of automorphisms of the graph of groups \mathcal{G} and a surjection $\pi : \text{Aut}(\mathcal{G}, \widehat{D}) \rightarrow C(\widehat{D})$. The maps from Theorem 5.10 are easy to define on $\text{Aut}(\mathcal{G}, \widehat{D})$, and the technical difficulties in Theorem 5.10 arise in showing that these homomorphisms factor through π .

This viewpoint works in cases more general than the one we consider in this paper. Our approach is close to that of Levitt [12], who gives a similar decomposition (to that of Theorem 5.10) of the automorphism group of a one-ended hyperbolic group by taking \mathcal{G} to be the graph of groups associated to the canonical JSJ decomposition of the hyperbolic group.

Cohen and Lustig also show that each Dehn twist in $\text{Out}(F_n)$ may be represented by an efficient Dehn twist. As each group $\text{Out}(G_w, \mathcal{C}_w)$ has a finite index subgroup with finite classifying space ([6], Corollary 6.1.4.), we have the following corollary:

Corollary 5.11. *If $\phi \in \text{Out}(F_n)$ is a Dehn twist automorphism then $C(\phi)$ has a finite-index, torsion-free subgroup with finite classifying space.*

In $\text{Aut}(F_n)$ the situation is slightly trickier, as here it is not true that every Dehn twist has an efficient representative. For this reason, in Section 6 we introduce the notion of a *pointedly efficient Dehn twist*. We show that every Dehn twist in $\text{Aut}(F_n)$ has a pointedly efficient representative, and for such an element we have the same decomposition as in Theorem 5.10. We use this in Section 7 to describe an algorithm to find a presentation of the centraliser of a Dehn twist automorphism in $\text{Aut}(F_n)$ or $\text{Out}(F_n)$. In Section 8 we use this algorithm to give a presentation for the centraliser of a Nielsen automorphism ρ in $\text{Aut}(F_n)$. This allows us to compute $H_1(C(\rho))$:

Corollary 8.3. *Let $\rho \in \text{Aut}(F_n)$ be a Nielsen automorphism. Then*

$$H_1(C(\rho)) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3, & \text{if } n = 3, \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{if } n = 4, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } n \geq 5. \end{cases}$$

When $n = 2$, the class $[\rho]$ is a primitive element of \mathbb{Z}^2 , when $n = 3$ it is twice a generator of \mathbb{Z} , and otherwise $[\rho] = 0$.

Corollary 8.3 has an application to actions of $\text{Aut}(F_n)$ on $\text{CAT}(0)$ spaces. With such actions, the centralisers of elements with non-zero *translation length* have infinite order in their abelianisation.

Corollary 8.6. *If $n \geq 4$, Nielsen automorphisms always act by zero translation length whenever $\text{Aut}(F_n)$ acts isometrically on a proper $\text{CAT}(0)$ space.*

This improves on a result of Bridson, who showed the above for $n \geq 6$. Furthermore, Bridson [3] describes actions of $\text{Aut}(F_3)$ on $\text{CAT}(0)$ spaces where Nielsen automorphisms have positive translation length. Hence the requirement that $n \geq 4$ in this corollary is as strong as possible.

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2. BACKGROUND

This section consists of background material on graphs of groups and their automorphisms. We take most of our notation from [5].

2.1. Graphs of groups. A *graph of groups* \mathcal{G} is a tuple

$$\mathcal{G} = \{\Gamma, (G_v)_{v \in V(\Gamma)}, (G_e)_{e \in E(\Gamma)}, (f_e)_{e \in E(\Gamma)}\}$$

such that:

- Γ is a finite, connected graph in the sense of Serre (cf. I §2.1 in [19]) with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$.
- Each G_e, G_v is a group.
- If $\tau(e)$ is the terminal vertex of an edge e , we have an injective *edge map* $f_e: G_e \rightarrow G_{\tau(e)}$.
- For any edge e , we have $G_e = G_{\bar{e}}$, where \bar{e} denotes the edge e with reversed orientation.

We let $\iota(e) = \tau(\bar{e})$ denote the initial vertex of an edge e .

2.2. The path group and related subsets. The *path group* of \mathcal{G} , denoted $\Pi(\mathcal{G})$, is defined by taking the free group F generated by the letters $(t_e)_{e \in E(\Gamma)}$ and quotienting out the free product $(\ast_{v \in V(\Gamma)} G_v) \ast F$ by the relations:

- $t_e = t_{\bar{e}}^{-1}$ for all $e \in E(\Gamma)$,
- $t_e f_e(a) t_e^{-1} = f_{\bar{e}}(a)$ for all $e \in E(\Gamma)$ and $a \in G_e$.

We say that an element $g \in \Pi(\mathcal{G})$ is connected if there exists a (possibly trivial) path e_1, \dots, e_k in Γ starting from a vertex v_0 and elements g_0, g_1, \dots, g_k such that $g_0 \in G_{v_0}$, $g_i \in G_{\tau(e_i)}$ for each $i \geq 1$ and:

$$g = g_0 t_{e_1} g_1 t_{e_2} \cdots g_{k-1} t_{e_k} g_k.$$

We define $\pi_1(\mathcal{G}, v, w)$ to be the set of elements of $\Pi(\mathcal{G})$ represented by connected words whose underlying paths start at v and end at w . If $v = w$, the set forms a group – the *fundamental group of the graph of groups* – and is denoted $\pi_1(\mathcal{G}, v)$.

Given any element x of a group G , let ad_x be the inner automorphism given by the map $g \mapsto xgx^{-1}$. In this paper, automorphisms always act on the left (so that $\text{ad}_{xy} = \text{ad}_x \text{ad}_y$). If $W \in \pi_1(\mathcal{G}, v, w)$ then the restriction of $\text{ad}_W: \Pi(\mathcal{G}) \rightarrow \Pi(\mathcal{G})$ to $\pi_1(\mathcal{G}, w)$ induces an isomorphism between $\pi_1(\mathcal{G}, w)$ and $\pi_1(\mathcal{G}, v)$.

2.3. Reduced words. The notion of reduced words in fundamental groups of graphs of groups may be extended to $\Pi(\mathcal{G})$ as follows:

Definition 2.1. A word $W = g_0 t_{e_1} g_1 t_{e_2} \cdots g_{k-1} t_{e_k} g_k$ is called *reduced* if $g_i \notin f_{e_i}(G_{e_i})$ whenever $e_{i+1} = \overline{e_i}$ for some $1 \leq i \leq k-1$.

Note that a word with $k \leq 1$ is always reduced. Sometimes we write t_i instead of t_{e_i} . Reduced words representing the same element of $\Pi(\mathcal{G})$ can only differ in the following way:

Proposition 2.2 ([5], Proposition 3.6). *Suppose we have two reduced words $V = r_0 t_1 r_1 \dots t_q r_q$ and $W = s_0 t'_1 s_1 \dots t'_q s_q$ representing the same element of $\Pi(\mathcal{G})$. Then:*

- $q = q'$ and $t_i = t'_i$ for all $i = 1, \dots, q$,
- there are $h_i \in G_{e_i}$ for all i , $1 \leq i \leq q$, such that

$$\begin{aligned} s_0 &= r_0 f_{\overline{e_1}}(h_1^{-1}), \\ s_i &= f_{e_i}(h_i) r_i f_{\overline{e_{i+1}}}(h_{i+1}^{-1}) \text{ for } 1 \leq i \leq q-1, \\ s_q &= f_{e_q}(h_q) r_q. \end{aligned}$$

In particular, we may define the *length* of an element of $\Pi(\mathcal{G})$ to be the length of the underlying path of any reduced word representing that element.

Every word W can be transformed to a reduced word representing the same element in $\Pi(\mathcal{G})$ using the defining relations of this group. During this procedure, the number of t -symbols strictly decreases in each step. Connected words are then transformed to connected, reduced words.

Lemma 2.3. *If V and W represent the same element in $\Pi(\Gamma)$, the word V is reduced, and the underlying paths of V and W have the same length, then W is reduced.*

Proof. If W was not reduced we could transform W to a reduced word of shorter length, which would be a contradiction to Proposition 2.2. \square

2.4. Automorphisms of graphs of groups. Let \mathcal{G} be a graph of groups. An *automorphism* of \mathcal{G} is a tuple of the form

$$\{H_\Gamma, (H_e)_{e \in E(\Gamma)}, (H_v)_{v \in V(\Gamma)}, (\delta(e))_{e \in E(\Gamma)}\},$$

where

- $H_\Gamma: \Gamma \rightarrow \Gamma$ is a graph isomorphism,
- $H_v: G_v \rightarrow G_{H_\Gamma(v)}$ is a group isomorphism,
- $H_e = H_{\bar{e}}: G_e \rightarrow G_{H_\Gamma(e)}$ is a group isomorphism,
- $\delta(e)$ is an element of $G_{\tau(H_\Gamma(e))}$,

with the additional compatibility requirement that

$$(1) \quad H_{\tau(e)}(f_e(a)) = \delta(e)f_{H_\Gamma(e)}(H_e(a))\delta(e)^{-1}$$

for all $e \in E(\Gamma)$ and $a \in G_e$. We shall sometimes look at the case where H_Γ is the identity on Γ and H_e is the identity map on each edge group. In this case, the compatibility requirement can be phrased in the simpler sense, that:

$$(2) \quad H_{\tau(e)}(f_e(a)) = \delta(e)f_e(a)\delta(e)^{-1}$$

for all edges $e \in E(\Gamma)$ and $a \in G_e$.

2.5. The automorphism group of \mathcal{G} . The set of automorphisms of \mathcal{G} forms a group, which we call $\text{Aut}(\mathcal{G})$. If H and H' are two automorphisms of \mathcal{G} , then their product $H'' = H.H'$ is defined as follows (for simplicity we write $H(v)$ and $H(e)$ instead of $H_\Gamma(v)$ and $H_\Gamma(e)$):

$$\begin{aligned} H''_\Gamma &= H_\Gamma H'_\Gamma, \\ H''_e &= H_{H'(e)} H'_e, \\ H''_v &= H_{H'(v)} H'_v, \\ \delta''(e) &= H_{H'(\tau(e))}(\delta'(e))\delta(H'(e)). \end{aligned}$$

This operation is associative and has an identity element where H_Γ, H_v, H_e are all the identity automorphisms and each $\delta(e)$ is trivial. Furthermore, each $H \in \text{Aut}(\mathcal{G})$ has an inverse H^{-1} defined by taking $(H^{-1})_\Gamma = (H_\Gamma)^{-1}$, $(H^{-1})_e = (H_{H_\Gamma^{-1}(e)})^{-1}$, $(H^{-1})_v = (H_{H_\Gamma^{-1}(v)})^{-1}$, and $\delta^{-1}(e) = H_{H^{-1}(\tau(e))}^{-1}(\delta(H^{-1}(e))^{-1})$.

2.6. The action of $\text{Aut}(\mathcal{G})$ on $\Pi(\mathcal{G})$. An element $H \in \text{Aut}(\mathcal{G})$ induces an automorphism $H_*: \Pi(\mathcal{G}) \rightarrow \Pi(\mathcal{G})$ by taking:

$$\begin{aligned} g &\mapsto H_v(g), & g &\in G_v, \\ t_e &\mapsto \delta(\bar{e})t_{H(e)}\delta(e)^{-1}, & t_e &\in F. \end{aligned}$$

The map H_* takes connected words to connected words, so for each vertex v of Γ there is an induced map $H_{*v}: \pi_1(\mathcal{G}, v) \rightarrow \pi_1(\mathcal{G}, H_\Gamma(v))$. If $H_\Gamma(v) = v$, then $H_{*v} \in \text{Aut}(\pi_1(\mathcal{G}, v))$. Similarly, if $H_\Gamma(v) = w$, we can choose an element $W \in \pi_1(\mathcal{G}, v, w)$ so that $\text{ad}_W H_{*v} \in \text{Aut}(\pi_1(\mathcal{G}, v))$. If $W, W' \in \pi_1(\mathcal{G}, v, w)$ then $\text{ad}_W H_{*v}$ and $\text{ad}_{W'} H_{*v}$ differ by $\text{ad}_{WW'^{-1}}$ in $\text{Aut}(\pi_1(\mathcal{G}, v))$, so H determines an element \hat{H} of $\text{Out}(\pi_1(\mathcal{G}, v))$.

Let $\text{Aut}(\mathcal{G}, v)$ to be the subgroup of $\text{Aut}(\mathcal{G})$ consisting of elements such that $H_\Gamma(v) = v$. The next lemma follows from the discussion above and the definition of multiplication in $\text{Aut}(\mathcal{G})$.

Lemma 2.4. *The map $H \mapsto \hat{H}$ induces a homomorphism*

$$U: \text{Aut}(\mathcal{G}) \rightarrow \text{Out}(\pi_1(\mathcal{G}, v))$$

*and the map $H \mapsto H_{*v}$ induces a homomorphism*

$$V: \text{Aut}(\mathcal{G}, v) \rightarrow \text{Aut}(\pi_1(\mathcal{G}, v)).$$

3. ELEMENTS OF $\text{Aut}(\mathcal{G})$ THAT ACT TRIVIALY ON $\pi_1(\mathcal{G}, v)$

We fix a graph of groups \mathcal{G} and a preferred vertex v . As our main aim is to study the action of $\text{Aut}(\mathcal{G})$ on $\pi_1(\mathcal{G}, v)$, we'd first like to understand when elements of $\text{Aut}(\mathcal{G})$ act trivially on $\pi_1(\mathcal{G}, v)$. Similar results to those contained in this section were obtained in [11] in the case of *free-cyclic* graphs of groups (see, for example, Corollary 4.9 of [11]).

Definition 3.1. Let $\text{KO}(\mathcal{G}) = \ker U$ be the subgroup of $\text{Aut}(\mathcal{G})$ consisting of elements such that $\hat{H} = 1$. Let $\text{KA}(\mathcal{G}) = \ker V$ be the subgroup of $\text{Aut}(\mathcal{G}, v)$ consisting of elements such that $H_{*v} = 1$.

In Proposition 3.4 we find generating sets of $\text{KO}(\mathcal{G})$ and $\text{KA}(\mathcal{G})$ in the case where no edge map f_e is surjective. When the edge and vertex groups of \mathcal{G} are finitely generated, we show that $\text{KO}(\mathcal{G})$ and $\text{KA}(\mathcal{G})$ are also finitely generated.

3.1. The action on the underlying graph Γ . Let $\text{Aut}(\Gamma)$ be the group of automorphisms of the underlying graph Γ . Let $\text{Aut}(\Gamma, v)$ be

the subgroup of $\text{Aut}(\Gamma)$ consisting of automorphisms that fix the vertex v . Then we have homomorphisms

$$\begin{aligned}\alpha: \text{Aut}(\mathcal{G}) &\rightarrow \text{Aut}(\Gamma) \\ \beta: \text{Aut}(\mathcal{G}, v) &\rightarrow \text{Aut}(\Gamma, v)\end{aligned}$$

given by taking $H \mapsto H_\Gamma$. We define $\text{Aut}^0(\mathcal{G}) = \ker \alpha = \ker \beta$.

Lemma 3.2. *Suppose that no edge map of \mathcal{G} is surjective. If $\widehat{H} = 1$, then $H_\Gamma(v) = v$, and there exists $g \in G_v$ such that $H_{*v} = \text{ad}_g$.*

Proof. Suppose $H \in \text{Aut}(\mathcal{G})$ and $\widehat{H} = 1$. Let W be an element of $\pi_1(\mathcal{G}, v, H(v))$. Then $\text{ad}_W H_{*v}$ is a representative of \widehat{H} in $\text{Out}(\pi_1(\mathcal{G}, v))$, and therefore, as $\widehat{H} = 1$, we have $\text{ad}_W H_{*v} = \text{ad}_V$ for some $V \in \pi_1(\mathcal{G}, v)$. In particular, $H_v(x) = W^{-1}VxV^{-1}W$ for each $x \in G_v$.

Let k denote the length of $W^{-1}V$. We claim that $k = 0$. This will show $W^{-1}V \in G_v$, so that $H(v) = v$ and $H_{*v} = \text{ad}_{W^{-1}V}$.

Suppose $k \geq 1$ and let $g_0 t_{e_1} g_1 \cdots t_{e_k} g_k$ be a reduced representative of $W^{-1}V$. As f_{e_k} is not surjective, we may choose $x \in G_v$ such that $g_k x g_k^{-1} \notin f_{e_k}(G_{e_k})$, so that $W^{-1}VxV^{-1}W$ is reduced and of length $2k > 0$ in $\Pi(\mathcal{G})$. However, $H_v(x)$ is of length 0, which is the desired contradiction. \square

We may now show that elements of $\text{Aut}(\mathcal{G})$ that act trivially on $\pi_1(\mathcal{G}, v)$ must also act trivially on Γ .

Proposition 3.3. *Suppose that no edge map f_e of \mathcal{G} is surjective. If $\widehat{H} = 1$, then $H_\Gamma = 1$. Hence $\text{KO}(\mathcal{G}) < \text{Aut}^0(\mathcal{G})$.*

Proof. Suppose that $\widehat{H} = 1$. Let e be an edge of Γ . There exists a reduced edge path from v to $\iota(e)$ or $\tau(e)$ that does not contain e or \bar{e} . Without loss of generality, assume we have a path e_1, \dots, e_k to $\iota(e)$ such that $e_k \neq \bar{e}$. Let $x \in G_{\tau(e)} \setminus f_e(G_e)$. Define $W = t_{e_1} \cdots t_{e_k} t_e x t_e^{-1} t_{e_k}^{-1} \cdots t_{e_1}^{-1}$. By Lemma 3.2, we have $H_\Gamma(v) = v$ and $H_{*v} = \text{ad}_g$ for some $g \in G_v$. As $W \in \pi_1(\mathcal{G}, v)$ we have:

$$H_{*v}(W) = g t_{e_1} \cdots t_{e_k} t_e x t_e^{-1} t_{e_k}^{-1} \cdots t_{e_1}^{-1} g^{-1}.$$

From the definition of H_* we also have a representative of $H_{*v}(W)$ as:

$$\delta(\bar{e}_1) t_{H(e_1)} \delta(e_1)^{-1} \cdots t_{H(e_k)} \delta(e_k)^{-1} H_{\tau(e)}(x) \delta(e) t_{H(e)}^{-1} \cdots \delta(e_1) t_{H(e_1)}^{-1} \delta(\bar{e}_1)^{-1}.$$

As both representatives are of the same length and the former representative of $H_*(W)$ is reduced, so is the latter by Lemma 2.3. Hence by comparing normal forms (Proposition 2.2), both representatives must have the same underlying edge path. In particular, $H_\Gamma(e) = e$. As our initial choice of edge was arbitrary, H_Γ fixes every edge of Γ . \square

3.2. The situation when $H_\Gamma = 1$. Given any vertex $w \in V(\Gamma)$ and $\gamma \in G_w$, we define the automorphism $M := M(w, \gamma)$ by

$$\begin{aligned} M_\Gamma &= 1, \\ M_e &= 1, \\ M_u &= \begin{cases} \text{ad}_\gamma & \text{if } u = w, \\ 1 & \text{if } u \neq w, \end{cases} \\ \delta_M(e) &= \begin{cases} \gamma & \text{if } \tau(e) = w, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

We have written δ_M to emphasize that we mean the δ corresponding to the graph of groups automorphism M . If the vertex w is equal to the basepoint v , then $M(v, \gamma)_{*v} = \text{ad}_\gamma$, otherwise $M(w, \gamma)_{*v} = 1$.

Next we fix some edge $e_0 \in \Gamma$ and $h \in G_{e_0}$. We define $K = K(e_0, h)$ in $\text{Aut}(\mathcal{G})$ by

$$\begin{aligned} K_\Gamma &= 1, \\ K_e &= \begin{cases} \text{ad}_h^{-1} & \text{if } e = e_0 \text{ or } e = \overline{e_0}, \\ 1 & \text{otherwise,} \end{cases} \\ K_u &= 1, \\ \delta_K(e) &= \begin{cases} f_e(h) & \text{if } e = e_0 \text{ or } e = \overline{e_0}, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

It is easily verified that $(K(e_0, h))_{*v} = 1$.

Proposition 3.4. *Suppose that no edge map f_e of \mathcal{G} is surjective. Then:*

- $\text{KO}(\mathcal{G})$ is generated by elements of the form $M(w, \gamma)$ and of the form $K(e, h)$.
- $\text{KA}(\mathcal{G})$ is generated by elements of the form $M(w, \gamma)$, with $w \neq v$ and of the form $K(e, h)$.

Proof. Suppose that $H \in \text{KO}(\mathcal{G})$, so that $\widehat{H} = 1$. By Lemma 3.2 and Proposition 3.3 we have $H_\Gamma = 1$, and there exists $g \in G_v$ such that $H_{*v} = \text{ad}_g$. Then $H_{*v}M(v, g^{-1})_{*v} = 1$, so after multiplying by $M(v, g^{-1})$ we may assume that $H_{*v} = 1$, and it is sufficient to prove the $\text{KA}(\mathcal{G})$ case of the proposition.

Let \mathcal{A} be the subgroup of $\text{Aut}(\mathcal{G})$ generated by the elements of the form $M(w, \gamma)$ (for $w \neq v$) and elements of the form $K(e, h)$. We define a subgraph $\Delta(H)$ of Γ , which may be thought of as the subgraph of Γ where H acts trivially. We say that a vertex w of Γ belongs to $\Delta(H)$ if

and only if $H_w = 1$. An edge e is defined to belong to $\Delta(H)$ if and only if its initial and terminal vertices are in $\Delta(H)$, and $\delta(e) = 1 = \delta(\bar{e})$.

As $H_{*v} = 1$ the vertex group automorphism $H_v = 1$, so $v \in \Delta(H)$. We define $\Lambda(H)$ to be the connected component of v in $\Delta(H)$.

Claim: If $\Lambda(H) \neq \Gamma$, then there is some $H' \in H.\mathcal{A}$ such that $\Lambda(H')$ strictly contains $\Lambda(H)$.

This claim will prove the proposition: we can apply it inductively to find H, H', H'', \dots such that $\Lambda(H) \subsetneq \Lambda(H') \subsetneq \Lambda(H'') \subsetneq \dots$. Since Γ is finite, we will eventually get some $\tilde{H} \in H.\mathcal{A}$ such that $\Lambda(\tilde{H}) = \Gamma$.

By applying the compatability condition (1) on page 6, we see that if $e \in \Lambda(H)$ then

$$f_e(a) = H_{\tau(e)}(f_e(a)) = f_e(H_e(a)).$$

As f_e is injective, we have $H_e = 1$. Then $\Lambda(H) = \Gamma$ ensures $\tilde{H} = 1$ and $H \in \mathcal{A}$, as asserted.

To prove the claim, assume that $\Lambda(H) \neq \Gamma$. Then, as Γ is connected, there is some edge $e \in \Gamma \setminus \Lambda(H)$ with initial vertex u lying in $\Lambda(H)$. Let $w = \tau(e)$.

Let e_1, \dots, e_k be an edge path from v to u in $\Lambda(H)$. Let $T := t_{e_1} \dots t_{e_k}$. Recall that $H_*(t_{e_i}) = \delta(\bar{e}_i)t_{e_i}\delta(e_i)^{-1}$. Since each e_i is an edge in $\Lambda(H) \subset \Delta(H)$, we have $\delta(e_i) = 1 = \delta(\bar{e}_i)$. Hence $H_*(t_{e_i}) = t_{e_i}$ and $H_*(T) = T$.

Pick some $g \in G_w \setminus f_e(G_e)$. Since $H_{*v} = 1$, we have $Tt_e g t_e^{-1} T^{-1} = H_*(Tt_e g t_e^{-1} T^{-1}) = TH_*(t_e g t_e^{-1})T^{-1}$ and so

$$\delta(\bar{e})t_e\delta(e)^{-1}H_w(g)\delta(e)t_e^{-1}\delta(\bar{e})^{-1} = t_e g t_e^{-1}.$$

As the right hand side of this equation is reduced, so is the left-hand side by Lemma 2.3. Therefore, by Proposition 2.2, there exist $h_1, h_2 \in G_e$ such that:

$$\begin{aligned} \delta(\bar{e}) &= f_{\bar{e}}(h_1^{-1}), \\ \delta(e)^{-1}H_w(g)\delta(e) &= f_e(h_1)gf_e(h_2^{-1}), \\ \delta(\bar{e})^{-1} &= f_{\bar{e}}(h_2). \end{aligned}$$

The first and third equations give rise to $\delta(\bar{e})^{-1} = f_{\bar{e}}(h_1) = f_{\bar{e}}(h_2)$. Since $f_{\bar{e}}$ is injective, we see that $h_1 = h_2$ and denote this element simply by h . By the second equation we obtain

$$H_w(g) = \delta(e)f_e(h)gf_e(h)^{-1}\delta(e)^{-1}$$

for all $g \in G_w \setminus f_e(G_e)$. Since f_e is not surjective, G_w is generated by such g and we conclude that $H_w = \text{ad}_{\delta(e)f_e(h)}$.

We need to consider the case when $w \notin \Lambda(H)$ and the case when $w \in \Lambda(H)$.

If $w \notin \Lambda(H)$, then we define $H' = H.M(w, f_e(h)^{-1}\delta(e)^{-1}).K(e, h)$. One can check that $\delta_{H'}(e)$, $\delta_{H'}(\bar{e})$ and H'_w are trivial. All other vertex automorphisms are unchanged, and $\delta_{H'}(e')$ differs from $\delta_H(e')$ only if $\tau(e') = w$. As $w \notin \Lambda(H)$, this doesn't affect any edges of $\Lambda(H)$. Thus $\Lambda(H') \supsetneq \Lambda(H)$.

If $w \in \Lambda(H)$, we choose an edge path $e'_1, \dots, e'_{k'}$ from w to v in $\Lambda(H)$. Let $T' := t_{e'_1} \dots t_{e'_{k'}}$. As for T , we have $H_*(T') = T'$. It follows that

$$T\delta(\bar{e})t_e\delta(e)^{-1}T' = H_{*v}(Tt_eT') = Tt_eT'$$

and hence $\delta(e) = t_e^{-1}\delta(\bar{e})t_e = t_e^{-1}f_{\bar{e}}(h^{-1})t_e = f_e(h^{-1})$. Then $H' = H.K(e, h)$ gives an automorphism for which $\Lambda(H) \subset \Lambda(H')$ and $e \in \Lambda(H')$. \square

As $M(u, \gamma)M(u, \gamma') = M(u, \gamma\gamma')$ and $K(e, h)K(e, h') = K(e, h'h)$, we have the following corollary:

Corollary 3.5. *If all vertex and edge groups of \mathcal{G} are finitely generated, then $\text{KO}(\mathcal{G})$ and $\text{KA}(\mathcal{G})$ are also finitely generated.*

Proposition 3.4 also implies that if two elements of $\text{Aut}(\mathcal{G})$ lie in the same coset of $\text{KO}(\mathcal{G})$, then their vertex maps can only differ by a series of inner automorphisms. Furthermore, if two elements are in the same coset of $\text{KA}(\mathcal{G})$, their automorphisms at v must agree.

Corollary 3.6. *Suppose that no edge map of \mathcal{G} is surjective. Then:*

- *Suppose that $\hat{H} = \hat{H}'$ and $H_\Gamma = H'_\Gamma = 1$. Then for every vertex w of Γ , the automorphisms H_w and H'_w are equal in $\text{Out}(G_w)$.*
- *Suppose further that $H_{*v} = H'_{*v}$. Then $H_v = H'_v$.*

4. DEHN TWISTS

4.1. Definition of Dehn twists.

Definition 4.1. An automorphism D of a graph of groups \mathcal{G} is called *Dehn twist* if

- $D_\Gamma = 1$,
- $D_w = 1$ for all $w \in V(\Gamma)$,
- $D_e = 1$ for all $e \in E(\Gamma)$,
- There are elements $\gamma_e \in Z(G_e)$ such that $\delta(e) = f_e(\gamma_e)$ for all $e \in E(\Gamma)$.

Every collection $(\gamma_e)_{e \in E(\Gamma)}$ with each $\gamma_e \in Z(G_e)$ defines a Dehn twist. To see this, we have to verify the compatibility condition (2) on

page 6. As D has trivial vertex group automorphisms and $\gamma_e \in Z(G_e)$, we have:

$$H_{\tau(e)}(f_e(a)) = f_e(a) = f_e(\gamma_e)f_e(a)f_e(\gamma_e)^{-1} = \delta(e)f_e(a)\delta(e)^{-1},$$

for any $a \in G_e$, as required. We say that the element $z_e = \gamma_e\gamma_{\bar{e}}^{-1}$ is the *twistor* of the edge e . It is easy to verify that Dehn twists form a subgroup of $\text{Aut}(\mathcal{G})$.

Definition 4.2. Let G be any group. An element $\phi \in \text{Aut}(G)$ or $\text{Out}(G)$ is a Dehn twist if there exists a graph of groups \mathcal{G} and an isomorphism $\rho : G \rightarrow \pi_1(\mathcal{G}, v)$ such that $\rho\phi\rho^{-1}$ is represented by a Dehn twist on \mathcal{G} .

Remark 4.3. Our definition of Dehn twist here coincides with the notion of Dehn twist in [5] defined by a set of twistors $(z_e)_{e \in E}$ such that each $z_e \in Z(G_e)$ and $z_e = z_{\bar{e}}^{-1}$. Conversely, if we are given a set of twistors $(z_e)_{e \in E(\Gamma)}$ such that each $z_e \in Z(G_e)$ and $z_{\bar{e}} = z_e^{-1}$, we may take an orientation E^+ of Γ (a subset of $E(\Gamma)$ such that for every edge e exactly one element of $\{e, \bar{e}\}$ lies in E^+), and define a Dehn twist in our sense by taking

$$\gamma_e = \begin{cases} z_e^{-1}, & \text{if } e \in E^+, \\ 1, & \text{if } e \notin E^+. \end{cases}$$

4.2. The subgroup of Dehn twists in $\text{Aut}(\mathcal{G})$.

Definition 4.4. Let $DA(\mathcal{G})$ and $DO(\mathcal{G})$ be the images of the subgroup of Dehn twists in $\text{Aut}(\mathcal{G})$ in $\text{Aut}(\pi_1(\mathcal{G}, v))$ and $\text{Out}(\pi_1(\mathcal{G}, v))$ respectively.

To look at these groups, we recall the following proposition from [5]:

Proposition 4.5 ([5], Proposition 5.4). *Let \mathcal{G} be a graph of groups with the property:*

(*) *for every edge $e \in E(\Gamma)$ there is an element $r_e \in G_e$ with*

$$f_e(G_e) \cap r_e f_e(G_e) r_e^{-1} = 1.$$

Then two Dehn twists $D = (\mathcal{G}, (\gamma_e)_{e \in E(\Gamma)})$, $D' = (\mathcal{G}, (\gamma'_e)_{e \in E(\Gamma)})$ with twistors $z_e = \gamma_e \gamma_{\bar{e}}^{-1}$ and $z'_e = \gamma'_e \gamma'_{\bar{e}}^{-1}$ determine the same outer automorphism of $\pi_1(\mathcal{G}, v)$ if and only if $z_e = z'_e$ for all $e \in E(\Gamma)$.

If each edge group $G_e \cong \mathbb{Z}$, then if we take an orientation E^+ of $E(\Gamma)$, we have $\mathbb{Z}^{|E^+|}$ choices for Dehn twists on \mathcal{G} with distinct image in $\text{Out}(\pi_1(\mathcal{G}, v))$. Multiplication of two Dehn twists is given by multiplying the twistors on each edge. Therefore Proposition 4.5 implies:

Proposition 4.6. *Let \mathcal{G} be a graph of groups with property $(*)$ such that each edge group $G_e \cong \mathbb{Z}$. Then $DA(\mathcal{G})$ and $DO(\mathcal{G})$ are free abelian groups of rank equal to the number of geometric (or unoriented) edges of Γ , i.e. the size of an orientation of $E(\Gamma)$.*

We will be looking at a particular class of graphs of groups which satisfy all the above hypotheses:

4.3. Efficient Dehn twists. Cohen and Lustig give a notion of when a Dehn twist is *efficient*. This may be thought of as when the graph of groups \mathcal{G} that the Dehn twist is defined on is, in a certain sense, optimal.

Definition 4.7. Let D be the Dehn twist given by $(\gamma_e)_{e \in E(\Gamma)}$. Two edges e' and e'' with common terminal vertex w are called

- *positively bonded*, if there are $m, n \geq 1$ such that $f_{e'}(z_{e'}^m)$ and $f_{e''}(z_{e''}^n)$ are conjugate in G_w ,
- *negatively bonded*, if there are $m \geq 1$ and $n \leq -1$ such that $f_{e'}(z_{e'}^m)$ and $f_{e''}(z_{e''}^n)$ are conjugate in G_w .

Definition 4.8 (cf. Definition 6.2 in [5]). A Dehn twist D given by $(\gamma_e)_{e \in E(\Gamma)}$ is called *efficient* if

- (1) Γ is *minimal*: If w has valence one and $w = \tau(e)$, then the edge map $f_e: G_e \rightarrow G_w$ is not surjective.
- (2) There is *no invisible vertex*: There is no 2-valent vertex w such that both edge maps $f_{e_i}: G_{e_i} \rightarrow G_w$ are surjective, where $\tau(e_1) = \tau(e_2) = w$ and $e_1 \neq e_2$.
- (3) *No unused edges*: For every edge e , we have $z_e \neq 0$, or equivalently $\gamma_{\bar{e}} \neq \gamma_e$.
- (4) *No proper powers*: If $r^p \in f_e(G_e)$ for some $p \neq 0$, then $r \in f_e(G_e)$.
- (5) Whenever $w = \tau(e_1) = \tau(e_2)$, then e_1 and e_2 are not *positively bonded*.

Suppose that $\pi_1(\mathcal{G}, v) \cong F_n$ and D is an efficient Dehn twist on \mathcal{G} . Then each edge and vertex group is a subgroup of F_n , so is also free. Condition (3) implies that each edge group has nontrivial center, as $z_e \in Z(G_e)$, therefore each edge group is infinite cyclic. Conditions (1) and (4) imply that if w is a valence one vertex then G_w is free of rank at least two. Similarly conditions (2) and (4) imply that if w is of valence two then G_w is free of rank at least two, and conditions (4) and (5) imply that if w is a vertex of valence at least three then G_w is also free of rank at least two. In particular, no edge map of \mathcal{G} is surjective,

and \mathcal{G} satisfies condition $(*)$ of Proposition 4.5 (see Proposition 6.4 of [5] for more detail).

5. CENTRALISERS OF EFFICIENT DEHN TWISTS OF FREE GROUPS

We shall now restrict ourselves to the case where $\pi_1(\mathcal{G}, v)$ is a free group of rank $n \geq 2$, and $D = (\mathcal{G}, (\gamma_e)_{e \in \Gamma})$ is an efficient Dehn twist defined on \mathcal{G} . We fix an isomorphism $F_n \cong \pi_1(\mathcal{G}, v)$ so that we may identify $\text{Aut}(\pi_1(\mathcal{G}, v))$ with $\text{Aut}(F_n)$.

5.1. Representing centralisers by abstract automorphisms. Let $C(\widehat{D})$ and $C(D_{*v})$ be the centralisers of \widehat{D} and D_{*v} in $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ respectively.

Definition 5.1. Let $\text{Aut}(\mathcal{G}, \widehat{D})$ be the subgroup of $\text{Aut}(\mathcal{G})$ consisting of elements H such that $H_e(z_e) = z_{H(e)}$ for every edge $e \in E$. Let $\text{Aut}(\mathcal{G}, D_{*v})$ be the subgroup of $\text{Aut}(\mathcal{G})$ consisting of elements H such that $H \in \text{Aut}(\mathcal{G}, \widehat{D})$, and $H_\Gamma(v) = v$.

Cohen and Lustig use the action of \widehat{D} on the boundary of Outer Space, where there are ‘rational’ points given by graph of groups decompositions of F_n , to show that elements of $C(\widehat{D})$ and $C(D_{*v})$ may be represented by elements of $\text{Aut}(\mathcal{G}, \widehat{D})$ and $\text{Aut}(\mathcal{G}, D_{*v})$ respectively. In particular, Proposition 7.1 of [5] may be rephrased as follows:

Proposition 5.2. *Let $D = (\mathcal{G}, (\gamma_e)_{e \in E(\Gamma)})$ be an efficient Dehn twist. The maps $H \mapsto \widehat{H}$ and $H \mapsto H_{*v}$ induce surjective homomorphisms:*

$$\begin{aligned} \text{Aut}(\mathcal{G}, \widehat{D}) &\rightarrow C(\widehat{D}), \\ \text{Aut}(\mathcal{G}, D_{*v}) &\rightarrow C(D_{*v}). \end{aligned}$$

We may therefore use the above subgroups of $\text{Aut}(\mathcal{G})$ to study the centralisers of efficient Dehn twists in $\text{Out}(F_n)$ and $\text{Aut}(F_n)$.

5.2. Evaluation on vertex group automorphisms. The centralisers of D in $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ act on the underlying graph of \mathcal{G} :

Proposition 5.3. *Let D be an efficient Dehn twist as above. The homomorphisms*

$$\begin{aligned} \alpha: \text{Aut}(\mathcal{G}, \widehat{D}) &\rightarrow \text{Aut}(\Gamma), \\ \beta: \text{Aut}(\mathcal{G}, D_{*v}) &\rightarrow \text{Aut}(\Gamma, v), \end{aligned}$$

given by $H \mapsto H_\Gamma$ descend to homomorphisms

$$\begin{aligned} \bar{\alpha}: C(\widehat{D}) &\rightarrow \text{Aut}(\Gamma), \\ \bar{\beta}: C(D_{*v}) &\rightarrow \text{Aut}(\Gamma, v). \end{aligned}$$

Proof. It is sufficient to show that if two elements of $\text{Aut}(\mathcal{G})$ agree in $\text{Out}(F_n)$ then they induce the same automorphism of Γ . Suppose that $\widehat{H} = \widehat{H}'$, so that $\widehat{H^{-1}H'} = 1$. If D is an efficient Dehn twist, then no edge map of \mathcal{G} is surjective, so by Proposition 3.3 we have $H_\Gamma^{-1}H'_\Gamma = (H^{-1}H')_\Gamma = 1$. Hence $H_\Gamma = H'_\Gamma$. \square

We define $\text{Aut}^0(\mathcal{G}, \widehat{D}), \text{Aut}^0(\mathcal{G}, D_{*v}), C^0(\widehat{D}), C^0(D_{*v})$ to be the kernels of the maps $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ respectively. As the images of α and β are finite groups, $C^0(\widehat{D})$ and $C^0(D_{*v})$ are finite index subgroups of $C(\widehat{D})$ and $C(D_{*v})$. Note that $\text{Aut}^0(\mathcal{G}, \widehat{D}) = \text{Aut}^0(\mathcal{G}, D_{*v})$, as both groups consist of automorphisms H that map twistors to twistors and act trivially on Γ .

As the edge groups of \mathcal{G} are cyclic, we may pick a generator a_e of each G_e .

Definition 5.4. Let \mathcal{C}_w be the set of conjugacy classes in G_w defined by:

$$\mathcal{C}_w = \{[f_e(a_e)] : e \in E(\Gamma), \tau(e) = w\}.$$

Let $\text{Aut}(G_w, \mathcal{C}_w)$ and $\text{Out}(G_w, \mathcal{C}_w)$ be the subgroups of $\text{Aut}(G_w)$ and $\text{Out}(G_w)$ respectively consisting of automorphisms that fix every conjugacy class in \mathcal{C}_w .

Lemma 5.5. Suppose that $H \in \text{Aut}^0(\mathcal{G}, \widehat{D})$.

- For every edge of Γ we have $H_e = 1$.
- For every vertex of Γ we have $H_w \in \text{Aut}(G_w, \mathcal{C}_w)$.

Proof. As $H \in \text{Aut}^0(\mathcal{G}, \widehat{D})$, we have $H_\Gamma = 1$. Also H preserves twistors, so for any edge we have $H_e(z_e) = z_e$. As each twistor is nontrivial, and each edge group is cyclic, this implies that $H_e = 1$. The consistency condition for elements of $\text{Aut}(\mathcal{G})$ (equation (2) on page 6) then implies:

$$H_{\tau(e)}(f_e(a)) = \delta(e)f_e(a)\delta(e)^{-1}.$$

Applying this equation over all edges such that $\tau(e) = w$ shows that H_w fixes every conjugacy class in \mathcal{C}_w , so lies in $\text{Aut}(G_w, \mathcal{C}_w)$. \square

We therefore have homomorphisms

$$\begin{aligned} A: \text{Aut}^0(\mathcal{G}, \widehat{D}) &\rightarrow \bigoplus_{w \in V(\Gamma)} \text{Out}(G_w, \mathcal{C}_w), \\ B: \text{Aut}^0(\mathcal{G}, \widehat{D}) &\rightarrow \text{Aut}(G_v, \mathcal{C}_v) \oplus \left(\bigoplus_{w \neq v} \text{Out}(G_w, \mathcal{C}_w) \right) \end{aligned}$$

induced by the mapping $H \mapsto (H_w)_{w \in V(\Gamma)}$.

Proposition 5.6. *The homomorphisms A and B descend to surjective homomorphisms*

$$\begin{aligned}\overline{A}: C^0(\widehat{D}) &\rightarrow \bigoplus_{w \in V(\Gamma)} \text{Out}(G_w, \mathcal{C}_w), \\ \overline{B}: C^0(D_{*v}) &\rightarrow \text{Aut}(G_v, \mathcal{C}_v) \oplus \left(\bigoplus_{w \neq v} \text{Out}(G_w, \mathcal{C}_w) \right).\end{aligned}$$

Proof. If $H, H' \in \text{Aut}^0(\mathcal{G}, \widehat{D})$ have the same image in $\text{Out}(F_n)$, then by Corollary 3.6, H_w and H'_w have the same image in $\text{Out}(G_w)$. Similarly Corollary 3.6 tells us that if H, H' have the same image in $\text{Aut}(F_n)$ then $H_v = H'_v$. Therefore A and B descend to the maps \overline{A} and \overline{B} above. It only remains to show that A and B are surjective. To do this, we take an element $H_w \in \text{Aut}(G_w, \mathcal{C}_w)$ for each vertex of Γ and build an element $H \in \text{Aut}^0(\mathcal{G}, \widehat{D})$ with H_w as the vertex automorphism at w . As $H_w \in \text{Aut}(G_w, \mathcal{C}_w)$, the conjugacy class of $f_e(a_e)$ is preserved by H_w , so there exists $\delta(e) \in G_w$ such that $H_w(f_e(a_e)) = \delta(e)f_e(a_e)\delta(e)^{-1}$. As a_e generates G_e , this identity holds for every element $a \in G_e$, and we may define H to be the automorphism of \mathcal{G} given by the trivial graph automorphism $H_\Gamma = 1$, the vertex automorphisms $(H_w)_{w \in V(\Gamma)}$, trivial edge automorphisms, and twisting factors $(\delta(e))_{e \in E(\Gamma)}$. \square

5.3. The kernels of \overline{A} and \overline{B} . We now study the kernels of the above maps. Recall that a subgroup U of a group G is called *malnormal* if, for any $u \in G$, we have $u \in U$ whenever $uUu^{-1} \cap U \neq 1$. When \mathcal{G} is the graph of groups associated to an efficient Dehn twist of a free group, the image of an edge group in a vertex group is a maximal cyclic subgroup ((3) of Definition 4.8), in particular:

Lemma 5.7. *For all edges e , the subgroup $f_e(G_e)$ is malnormal in the vertex group $G_{\tau(e)}$.*

The following proposition is valid for general Dehn twists, although we are mostly interested in efficient Dehn twists of free groups.

Proposition 5.8. *Suppose $f_e(G_e)$ is non-trivial and malnormal in $G_{\tau(e)}$ for all edges e . Let $H \in \text{Aut}(\mathcal{G})$ such that $H_\Gamma = 1$, $H_e = 1$ for all e and that all H_w , $w \in V(\Gamma)$ are inner automorphisms of G_w . Then H is a composition of a Dehn twist and some maps $M(w, \gamma)$. If $H_v = 1$, then no $M(v, \gamma)$ is needed.*

Proof. Clearly all $M(w, \gamma)$ satisfy the conditions of the statement. Multiplying by all $M(w, x_w^{-1})$, where $H_w = \text{ad}_{x_w}$, we may assume that all H_w are the identity. Since these automorphisms are trivial on edge groups, the requirement $H_e = 1$ is respected.

Equation (2) on page 6 then simplifies to

$$f_e(a) = \delta(e)f_e(a)\delta(e)^{-1}$$

for all $e \in E(\Gamma)$ and $a \in G_e$. In particular, the inner automorphism $\text{ad}_{\delta(e)}$ of $G_{\tau(e)}$ preserves the image $f_e(G_e)$. Since we assume that this is malnormal, $\delta(e) = f_e(\gamma_e)$ for some $\gamma_e \in G_e$. But then H is a Dehn twist. \square

A result similar to the following has been obtained in Proposition 2.2 of [12].

Corollary 5.9. *Every element of $\ker \bar{A}$ and $\ker \bar{B}$ is represented by a Dehn twist on \mathcal{G} . Conversely, any element of $\text{Out}(F_n)$ or $\text{Aut}(F_n)$ represented by a Dehn twist on \mathcal{G} lies in $\ker \bar{A}$ or $\ker \bar{B}$ respectively. Hence $\ker \bar{A} = DO(\mathcal{G})$ and $\ker \bar{B} = DA(\mathcal{G})$.*

Proof. If $\phi \in \ker \bar{A}$ then ϕ is represented by an element $H \in \text{Aut}^0(\mathcal{G}, \hat{D})$ such that $H_\Gamma = 1$ and $H_e = 1$ for every edge of Γ by Lemma 5.5. Furthermore $H \in \ker A$, so that H_w is inner for every vertex of Γ . Hence H satisfies the hypothesis of Proposition 5.8. As $\widehat{M(w, \gamma)} = 1$ for every vertex $w \in V(\Gamma)$, Proposition 5.8 implies that ϕ is represented by a Dehn twist on \mathcal{G} . If $\Phi \in \ker \bar{B}$ then Φ is represented by an element H satisfying all of the above as well as having $H_v = 1$. As $M(w, \gamma)_{*v} = 1$ if $w \neq v$, Proposition 5.8 implies that Φ is also represented by a Dehn twist on \mathcal{G} .

Conversely, if D' is any Dehn twist on \mathcal{G} then $D' \in \text{Aut}^0(\mathcal{G}, \hat{D})$ and $D'_w = 1$ for every vertex of Γ . Therefore $\hat{D}' \in \ker \bar{A}$ and $D'_{*v} \in \ker \bar{B}$. \square

5.4. A short exact sequence for $C^0(\hat{D})$ and $C^0(D_{*v})$. We are now in a position to prove our main theorem:

Theorem 5.10. *Let D be an efficient Dehn twist on a graph of groups \mathcal{G} with $\pi_1(\mathcal{G}, v) \cong F_n$. Let $C(\hat{D})$ and $C(D_{*v})$ be the centralisers of D in $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ respectively. There exist homomorphisms:*

$$\begin{aligned} \bar{\alpha}: C(\hat{D}) &\rightarrow \text{Aut}(\Gamma), \\ \bar{\beta}: C(D_{*v}) &\rightarrow \text{Aut}(\Gamma, v), \end{aligned}$$

such that their kernels $C^0(\widehat{D})$ and $C^0(D_{*v})$ are finite index subgroups that fit into the exact sequences:

$$1 \rightarrow DO(\mathcal{G}) \rightarrow C^0(\widehat{D}) \rightarrow \bigoplus_{w \in V(\Gamma)} \text{Out}(G_w, \mathcal{C}_w) \rightarrow 1,$$

$$1 \rightarrow DA(\mathcal{G}) \rightarrow C^0(D_{*v}) \rightarrow \text{Aut}(G_v, \mathcal{C}_v) \oplus \left(\bigoplus_{w \neq v} \text{Out}(G_w, \mathcal{C}_w) \right) \rightarrow 1,$$

where $DA(\mathcal{G})$ and $DO(\mathcal{G})$ are free abelian groups of Dehn twists of rank equal to the number of geometric edges of \mathcal{G} .

Proof. The maps $\bar{\alpha}$ and $\bar{\beta}$ are given by Proposition 5.3. As $\text{Aut}(\Gamma)$ is finite, $C^0(\widehat{D})$ and $C^0(D_{*v})$ are finite index subgroups of $C(\widehat{D})$ and $C(D_{*v})$ respectively. The existence, as well as surjectivity, of the right hand maps (\bar{A} and \bar{B}) in our exact sequences is shown by Proposition 5.6. Corollary 5.9 then tells us that $\ker \bar{A} = DO(\mathcal{G})$ and $\ker \bar{B} = DA(\mathcal{G})$. By Proposition 4.6 the groups $DO(\mathcal{G})$ and $DA(\mathcal{G})$ are free abelian groups of rank equal to the number of geometric edges of Γ . \square

5.5. Finiteness properties. Every Dehn twist in $\text{Out}(F_n)$ may be represented by an efficient Dehn twist D ([5], Section 8.2.). Each group $\text{Out}(G_w, \mathcal{C}_w)$ in the exact sequence for $C^0(\widehat{D})$ has a finite-index torsion-free subgroup A_w with finite classifying space ([6], Corollary 6.1.4.). Let H be the intersection of the preimages of A_w in $C^0(\widehat{D})$. Then H fits in the exact sequence

$$1 \rightarrow DO(\mathcal{G}) \rightarrow H \rightarrow \bigoplus_{w \in V(\Gamma)} A_w \rightarrow 1,$$

and, as both ends of this exact sequence have finite classifying spaces, so does H (see, for example, Theorem 7.1.1. of [8]).

Corollary 5.11. *If $\phi \in \text{Out}(F_n)$ is a Dehn twist automorphism then $C(\phi)$ has a finite-index, torsion-free subgroup with finite classifying space.*

In particular, the centraliser of a Dehn twist automorphism in $\text{Out}(F_n)$ is finitely presented. In Section 7 we prove that the process of finding a presentation for the centraliser of a Dehn twist in $\text{Aut}(F_n)$ or $\text{Out}(F_n)$ can be made algorithmic.

6. POINTEDLY EFFICIENT DEHN TWISTS

It is shown in Section 8 of [5] that any Dehn twist in $\text{Out}(F_n)$ is represented by an efficient Dehn twist automorphism. This is not the case with $\text{Aut}(F_n)$. In this section we introduce the notion of a *pointedly*

efficient Dehn twist, and show that every Dehn twist in $\text{Aut}(F_n)$ is represented by a pointedly efficient Dehn twist. We then show that the centraliser of such an element satisfies the same exact sequences as in Theorem 5.10.

6.1. Building efficient representatives. In [5], Cohen and Lustig describe transformations of a Dehn twist D on a graph of groups \mathcal{G} . We briefly describe them here and refer the reader to Definition 8.2 of [5] for the precise definitions:

- (M1) *Transition to a proper subgraph:* Remove a valence one vertex w when its corresponding edge map is surjective.
- (M2) *Delete an invisible vertex with negatively bonded edges:* Remove a valence two vertex w when both edge maps are surjective and negatively bonded.
- (M3) *Fold positively bonded edges:* Fold two positively bonded edges e and e' at a vertex w .
- (M4) *Contract unused edges:* Collapse an edge e with trivial twistor and replace the vertex group(s) with an HNN extension or an amalgam, depending on whether e was a loop or not.
- (M5) *Get rid of proper powers:* Adjoin a formal root to an edge group when $f_e(a_e)$ is a proper power.

If D is a Dehn twist on \mathcal{G} that fails (1), (2), (3), (4), or (5) in the definition of efficient (Definition 4.8) then we may apply one of the moves (M1)–(M5) to obtain a new Dehn twist D' on a graph of groups \mathcal{G}' . Furthermore:

Lemma 6.1 (Lemma 8.3, [5]). *For any of the operations (M1)–(M5) there exist vertices $w \in V(\Gamma)$ and $w' \in V(\Gamma')$ and an isomorphism $\rho : \pi_1(\mathcal{G}, w) \rightarrow \pi_1(\mathcal{G}', w')$ such that $\widehat{D}' = \rho \widehat{D} \rho^{-1}$.*

In the $\text{Aut}(F_n)$ case, if we perform a move (M1) or (M2) when the vertex removed is equal to our chosen basepoint v , this may cause problems (we cannot move the basepoint). For this reason we define:

- (M1*) Perform (M1) at a vertex $w \neq v$.
- (M2*) Perform (M2) at a vertex $w \neq v$.

Lemma 6.2. *Let D be a Dehn twist on a graph of groups \mathcal{G} with basepoint v . For any of the operations (M1*), (M2*), (M3)–(M5) there exists a vertex $v' \in V(\Gamma')$ and an isomorphism $\rho : \pi_1(\mathcal{G}, v) \rightarrow \pi_1(\mathcal{G}', v')$ such that $D'_{*v} = \rho D_{*v} \rho^{-1}$.*

Proof. This follows from the proof of Lemma 8.3 of [5]. The isomorphism ρ may be chosen so that D' represents the same element of

$\text{Aut}(F_n)$ (rather than $\text{Out}(F_n)$) as long as we do not remove the base vertex using moves (M1) or (M2). \square

6.2. Pointedly efficient Dehn twists. The work above tells us that we need to be more careful when improving representatives of Dehn twists in $\text{Aut}(F_n)$. This leads to the following definition:

Definition 6.3. Let D be a Dehn twist on a graph of groups \mathcal{G} with a chosen basepoint v . It is called *pointedly efficient* if

- (1*) Γ is *minimal away from the basepoint*: if $w \neq v$ has valence one and $w = \tau(e)$, then the edge map f_e is not surjective,
- (2*) there is *no invisible vertex away from the basepoint*: There is no 2-valent vertex $w \neq v$ such that both edge maps f_{e_i} are surjective, where $\tau(e_1) = \tau(e_2) = w$ and $e_1 \neq e_2$,

and the conditions (3)–(5) of Definition 4.8 are satisfied.

Proposition 8.4 of [5] tells us that for any Dehn twist one can iteratively apply the operations (M1)–(M5) only a finite number of times. A Dehn twist is pointedly efficient if we are unable to apply any of the moves (M1*), (M2*), (M3)–(M5). Hence:

Proposition 6.4. *Iteratively applying moves (M1*), (M2*), (M3)–(M5) gives an algorithm to obtain a pointedly efficient representative from any Dehn twist representative of an element of $\text{Aut}(F_n)$.*

Most of our previous work on automorphisms of graphs of groups required that no edge map of \mathcal{G} was surjective. As this is not always the case with pointedly efficient Dehn twists, we need to work a little harder to obtain exact sequences similar to those of Theorem 5.10 for these elements.

6.3. Stabilisation. Given a graph of groups \mathcal{G} with chosen basepoint v , the *stabilisation* \mathcal{G}' of \mathcal{G} is the graph of groups obtained as follows: The underlying graph Γ is the same. G_v is replaced by $G'_v = G_v * \mathbb{Z}$, whereas the other vertex groups and all edge groups are not modified. There is an obvious inclusion $i: G_v \rightarrow G'_v$. We define $f'_e = f_e$ if $\tau(e) \neq v$, otherwise define $f'_e = i \circ f_e$. The injection i then induces stabilisation maps:

$$\begin{aligned} i_{G_v}: \text{Aut}(G_v) &\rightarrow \text{Aut}(G'_v), \\ i_{F_n}: \text{Aut}(\pi_1(\mathcal{G}, v)) &\rightarrow \text{Aut}(\pi_1(\mathcal{G}', v)) \cong \text{Aut}(F_{n+1}), \\ i_{\mathcal{G}}: \text{Aut}(\mathcal{G}, v) &\rightarrow \text{Aut}(\mathcal{G}', v). \end{aligned}$$

The first two maps are defined by extending the relevant automorphism to act trivially on the new free factor. $i_{\mathcal{G}}(H) = H'$ is defined to

be the automorphism of \mathcal{G}' with $H'_v = i_{G_v}(H_v)$ and $\delta'(e) = i(\delta(e))$ if $\tau(H_\Gamma(e)) = v$. The remaining data is the same as for H . Given a Dehn twist $D = (\mathcal{G}, (\gamma_e)_{e \in E(\Gamma)})$ of \mathcal{G} , the same elements γ_e define a Dehn twist D' on \mathcal{G}' . Then $i_{\mathcal{G}}(D) = D'$ and $i_{F_n}(D_{*v}) = D'_{*v}$. This lemma follows directly from the definitions:

Lemma 6.5. *D is pointedly efficient if and only if its stabilisation D' is efficient.*

6.4. The image of $C(D_{*v})$ in $C(D'_{*v})$. Suppose that D is pointedly efficient and D' is the corresponding efficient Dehn twist on the stabilised graph \mathcal{G}' . As we have a good understanding of $C(D'_{*v})$, we may understand $C(D_{*v})$ by looking at its image in $C(D'_{*v})$ under i_{F_n} .

Proposition 6.6. *$\phi' \in i_{F_n}(C(D_{*v}))$ if and only if ϕ' has a representative $H' \in \text{Aut}(\mathcal{G}', D'_{*v})$ such that $H'_v = i_{G_v}(H_v)$ for some $H_v \in \text{Aut}(G_v)$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \text{Aut}(\mathcal{G}, D_{*v}) & \longrightarrow & C(D_{*v}) \\ \downarrow i_{\mathcal{G}} & & \downarrow i_{F_n} \\ \text{Aut}(\mathcal{G}', D'_{*v}) & \longrightarrow & C(D'_{*v}) \end{array}$$

where the vertical maps are given by stabilisation and the horizontal maps by $H \mapsto H_{*v}$. The vertical maps are clearly injective.

Suppose that $\phi' = i_{F_n}(\phi)$ for some $\phi \in C(D_{*v})$. As D' is efficient, the lower horizontal map is surjective by Proposition 5.2, and there exists $H' \in \text{Aut}(\mathcal{G}', D'_{*v})$ such that $H'_{*v} = \phi'$. Then

$$H'_v = \phi'|_{G_v * \mathbb{Z}} = i_{G_v}(\phi|_{G_v}),$$

which completes the ‘only if’ part of this proposition.

Conversely, suppose that ϕ' has a representative $H' \in \text{Aut}(\mathcal{G}', D'_{*v})$ such that $H'_v = i_{G_v}(H_v)$ for some $H_v \in \text{Aut}(G_v)$. It is enough to show that $H' = i_{\mathcal{G}}(H)$ for some $H \in \text{Aut}(\mathcal{G}, D_{*v})$, by commutativity of the above square. We can build such a H as long as $\delta_{H'}(e) \in G_v$ for each edge e with $\tau(e) = v$. Suppose we have such an edge e with edge group generated by a . Then by the compatibility condition for automorphisms of graphs of groups:

$$\begin{aligned} \delta_{H'}(e) f'_{H'(e)}(H_e(a)) \delta_{H'}(e)^{-1} &= H'_v(f'_e(a)) \\ &= i_{G_v}(H_v)(f'_e(a)). \end{aligned}$$

As $f'_e = f_e \circ i$ and $f'_{H'(e)} = f_{H'(e)} \circ i$, where f_e and $f_{H'(e)}$ are the edge maps in \mathcal{G} , the elements $i_{G_v}(H_v)(f'_e(a_e))$ and $f'_{H'(e)}(H'_e(a_e))$ lie in $G_v \setminus \{1\}$. As G_v is malnormal in G'_v this implies $\delta_{H'}(e)$ lies in G_v also. \square

6.5. The pointed version of the exact sequence. We are now ready to prove the pointed analogue of Theorem 5.10:

Theorem 6.7. *Let D be a pointedly efficient Dehn twist on a graph of groups \mathcal{G} . Then there is a homomorphism $\bar{\beta} : C(D_{*v}) \rightarrow \text{Aut}(\Gamma, v)$ such that its kernel $C^0(D_{*v})$ is a finite index subgroup fitting into the exact sequence*

$$1 \rightarrow DA(\mathcal{G}) \rightarrow C^0(D_{*v}) \xrightarrow{\bar{\beta}} \text{Aut}(G_v, \mathcal{C}_v) \oplus \left(\bigoplus_{w \neq v} \text{Out}(G_w, \mathcal{C}_w) \right) \rightarrow 1,$$

where $DA(\mathcal{G})$ is a free abelian group of rank equal to the number of geometric edges of Γ .

Proof. Let $\bar{\beta}' : C(D'_{*v}) \rightarrow \text{Aut}(\Gamma, v)$ be the map given by Proposition 5.3. Define $\bar{\beta} = \bar{\beta}' \circ i_{F_n}$ and $C^0(D_{*v}) = \ker(\bar{\beta})$. We may view $C^0(D_{*v})$ as a subgroup of $C^0(D'_{*v})$ via the map i_{F_n} . Let $\mathcal{C}_w, \mathcal{C}'_w$ be the sets of conjugacy classes in F_n and F_{n+1} as defined in Definition 5.4. As D' is efficient we have an exact sequence

$$DA(\mathcal{G}') \hookrightarrow C^0(D'_{*v}) \xrightarrow{\bar{B}'} \text{Aut}(G'_v, \mathcal{C}'_v) \oplus \left(\bigoplus_{w \neq v} \text{Out}(G'_w, \mathcal{C}'_w) \right)$$

given to us by Theorem 5.10, which induces the exact sequence

$$DA(\mathcal{G}') \cap i_{F_n}(C^0(D_{*v})) \hookrightarrow i_{F_n}(C^0(D_{*v})) \xrightarrow{\bar{B}'} \bar{B}' \circ i_{F_n}(C^0(D_{*v})).$$

Each element of $DA(\mathcal{G}')$ is represented by an element $H \in \text{Aut}(\mathcal{G}', D'_{*v})$ such that $H_w = 1$ for every vertex w . Therefore by Proposition 6.6 we have $DA(\mathcal{G}') \cap i_{F_n}(C^0(D_{*v})) = DA(\mathcal{G}')$, a free abelian group of rank equal to the number of geometric edges of Γ . As the map \bar{B}' is induced by taking $H \mapsto H_w$ for every vertex w , it also follows from Proposition 6.6 that the image of $\bar{B}' i_{F_n}$ is isomorphic to $\text{Aut}(G_v, \mathcal{C}_v) \oplus \left(\bigoplus_{w \neq v} \text{Out}(G_w, \mathcal{C}_w) \right)$. \square

Remark 6.8. Chasing the definitions, it can be shown that the maps $\bar{\beta}$ and \bar{B} in Theorem 6.7 can be defined in the same manner as the maps of the same names in Proposition 5.3 and Proposition 5.6.

Remark 6.9. Corollary 5.11 is also valid for a Dehn twist $\phi \in \text{Aut}(F_n)$. To see this, we first make ϕ pointedly efficient. Then we use the exact sequence $1 \rightarrow G_v \rightarrow \text{Aut}(G_v, \mathcal{C}_v) \rightarrow \text{Out}(G_v, \mathcal{C}_v) \rightarrow 1$ to find a finite

index subgroup of $\text{Aut}(G_v, \mathcal{C}_v)$ with a finite classifying space. Then we proceed as in Subsection 5.5.

7. COMPUTING FINITE PRESENTATIONS

In this section we describe how we can algorithmically determine a presentation for the centraliser of a Dehn twist automorphism in $\text{Aut}(F_n)$ or $\text{Out}(F_n)$.

7.1. Ingredients. First we recall algorithms which we need as ingredients.

- **The McCool complex.** In [15], McCool describes an algorithm to build a 2-dimensional finite CW complex whose fundamental group can be identified with $\text{Out}(F_n, \mathcal{C})$ for any set \mathcal{C} of conjugacy classes. This clearly leads to an algorithm providing a finite presentation for such a group.
- **Whitehead's algorithm.** ([13], Proposition I.4.21.) If we have two sets of conjugacy classes $\alpha_1, \dots, \alpha_t$ and $\alpha'_1, \dots, \alpha'_t$ in F_n then it is decidable whether there is an automorphism $\phi \in \text{Aut}(F_n)$ such that $\phi(\alpha_1) = \alpha'_1, \phi(\alpha_2) = \alpha'_2, \dots, \phi(\alpha_t) = \alpha'_t$.
- **Short exact sequences.** (Proposition 7.1 below) Given a short exact sequence $1 \rightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \rightarrow 1$, finite presentations of A and C , and a reasonable knowledge of B (e.g. a generating set for B as a subgroup of a finitely presented group), we may find a finite presentation of B .

Let us go into detail about what we need to assume for this final bullet point. We are given a finite presentation of A with generating set $X = \{x_1, \dots, x_k\}$ and set of relations R and a finite presentation of C with generating set $Z = \{z_1, \dots, z_m\}$ and relations S . Suppose that we can find $\tilde{z}_1, \dots, \tilde{z}_m$ such that $\pi(\tilde{z}_j) = z_j$. Taking the union of $\iota(X) = \{\iota(x_1), \dots, \iota(x_k)\}$ and $\tilde{Z} = \{\tilde{z}_1, \dots, \tilde{z}_m\}$ gives a generating set for B .

We need three types of relations. We first have the relations in B given by R , the set of relations in A . We call these the *kernel relations*.

We may lift the set of relations S to a set of words \tilde{S} on \tilde{Z} in the obvious way. Each element of \tilde{S} is mapped to the identity under π , so lies in $\iota(A)$. Suppose that for each element $\tilde{s} \in \tilde{S}$ we may find a word w_s in $\iota(A)$ such that $w_s = \tilde{s}$ in B . We call the set $\{w_s = \tilde{s} : s \in S\}$ the set of *lifted relations*.

Finally note that for $\epsilon \in \{1, -1\}$ the element $\tilde{z}_j^\epsilon \iota(x_i) \tilde{z}_j^{-\epsilon}$ is mapped to 1 by π . Our final assumption is that we may find a word $w_{i,j,\epsilon}$ in $\iota(A)$ such that $w_{i,j,\epsilon} = \tilde{z}_j^\epsilon \iota(x_i) \tilde{z}_j^{-\epsilon}$ in B . We say the set of all such relations

is called the set of *conjugation relations*. The following is an exercise in combinatorial group theory:

Proposition 7.1. *B has a finite presentation given by the generating set $\iota(X) \sqcup \tilde{Z}$ with the kernel relations, lifted relations and conjugation relations.*

In this paper B will always be a subgroup of $\text{Aut}(F_n)$ or $\text{Out}(F_n)$, and the map π will be well-understood. In the cases we consider it is always possible to find the lifts $\tilde{z}_1, \dots, \tilde{z}_m$ algorithmically. Furthermore, words of the form w_s or $w_{i,j,\epsilon}$ may be found by running through all words in $\iota(A)$ until we find one equal to \tilde{s} or $\tilde{z}_j^\epsilon \iota(x_i) \tilde{z}_j^{-\epsilon}$ respectively.

7.2. Algorithm for a presentation of the centraliser. We are now in a position to describe the algorithm. Given a Dehn twist representative for some $\phi \in \text{Aut}(F_n)$ or $\text{Out}(F_n)$, one proceeds as follows:

Step 1. Convert the Dehn twist to a (pointedly) efficient representative as in Section 6.

Step 2. Find a presentation of the right hand term in the first short exact sequence in Theorem 5.10 or that in Theorem 6.7 by means of the McCool complex for each summand.

Step 3. Use the respective short exact sequence to determine a presentation of $C^0(\hat{D})$ or $C^0(D_{*v})$ as in Proposition 7.1 and the remark following it.

Step 4. Find a presentation for the image of $\bar{\alpha}$ or $\bar{\beta}$ in $\text{Aut}(\Gamma)$: One must find the different possibilities for H_Γ of arbitrary elements H in $\text{Aut}(\mathcal{G}, \hat{D})$ or $\text{Aut}(\mathcal{G}, D_{*v})$. One proceeds as follows: for each edge fix a generator $a_e \in G_e \cong \mathbb{Z}$ and $n_e > 0$ such that $z_e = a_e^{n(e)}$. Since $z_{\bar{e}} = z_e^{-1}$, we get $a_{\bar{e}} = a_e^{-1}$ and $n(\bar{e}) = n(e)$ for every edge e . In the Out case, a graph isomorphism $h \in \text{Aut}(\Gamma)$ is in the image of $\bar{\alpha}$ if and only if $n_{h(e)} = n_e$ for all e , and for each vertex w there is an isomorphism from G_w to $G_{h(w)}$ mapping each conjugacy class $[f_e(a_e)]$ in G_w to the conjugacy class $[f_{h(e)}(a_{h(e)})]$ in $G_{h(w)}$. To check this, one first checks whether the ranks of the free groups agree and, if so, one uses Whitehead's algorithm as stated above. In the Aut case, the image of $\bar{\beta}$ is determined in the same way, but only for graph automorphisms fixing the basepoint v .

Pick a finite presentation for this subgroup of $\text{Aut}(\Gamma)$, e.g. take all group elements as generators and the obvious relations given by group multiplication.

Step 5. Compute a presentation of the centraliser $C(\widehat{D})$ or $C(D_{*v})$ as in Proposition 7.1 using the exact sequence

$$1 \rightarrow C^0(\widehat{D}) \rightarrow C(\widehat{D}) \xrightarrow{\bar{\alpha}} \text{im } \bar{\alpha} \rightarrow 1$$

or

$$1 \rightarrow C^0(D_{*v}) \rightarrow C(D_{*v}) \xrightarrow{\bar{\beta}} \text{im } \bar{\beta} \rightarrow 1.$$

Remark 7.2. Although this is an algorithm to compute an explicit finite presentation, the McCool complex usually has a huge number of cells. Hence it is hard to write down the resulting presentation of the centraliser by hand. Therefore it is desirable to simplify these presentations. For the stabiliser of conjugacy classes of basis elements this has been done in [10], and the first author describes simplified presentations for stabilisers of more general elements in [18].

8. CENTALISER OF A NIELSEN AUTOMORPHISM

In this section we apply the work from the rest of the paper to Nielsen automorphisms, which have particularly simple Dehn twist representatives. We give a presentation for the centraliser of a Nielsen automorphism, use this presentation to compute the abelianisation of the centraliser, and finally describe how this computation restricts actions of $\text{Aut}(F_n)$ on $\text{CAT}(0)$ spaces.

8.1. Nielsen automorphisms of F_n as Dehn twists. Let Γ be the graph with one vertex v and one loop e (that is two oriented edges e and \bar{e}). We take G_v to be a free group with basis $B, b, c_1, \dots, c_{n-2}$. Let G_e be infinite cyclic with generator r . The edge maps are defined by $f_e(r) = b$ and $f_{\bar{e}}(r) = B$.

The fundamental group $\pi_1(\mathcal{G}, v)$ is the full path group $\Pi(\mathcal{G})$ here. It is generated by $t_e =: a, B, b$, and the c_i subject to the relation $aba^{-1} = B$. In other words, it is the free group with basis $a, b, c_1, \dots, c_{n-2}$. We define a Dehn twist D by $\gamma_e = r^{-1}$ and $\gamma_{\bar{e}} = 1$. Then $\delta(e) = f_e(\gamma_e) = b^{-1}$ and $\delta(\bar{e}) = f_{\bar{e}}(\gamma_{\bar{e}}) = 1$. It follows that D_{*v} maps a to ab and fixes b and all c_i . Hence $\rho := D_{*v}$ is a Nielsen automorphism. Note that D is efficient.

We may now compute an explicit presentation of the centraliser of ρ by the algorithm outlined in section 7. As $DA(\mathcal{G})$ is infinite cyclic and generated by ρ , the short exact sequence for $C^0(\rho)$ simplifies to

$$(3) \quad 1 \rightarrow \langle \rho \rangle \rightarrow C^0(\rho) \rightarrow \text{Aut}(G_v, \mathcal{C}_v) \rightarrow 1,$$

where $\mathcal{C}_v = \{[b], [B]\}$ is a set of two conjugacy classes of basis elements. Hence the first item we need is a presentation of $\text{Aut}(G_v, \mathcal{C}_v)$. This was found in [10], and we will review it in Proposition 8.1 below.

8.2. A presentation for $\text{Aut}(G_v, \mathcal{C}_v)$. In the following, we use $P_{i,j}$ to denote the automorphism of either $F_n = \langle a, b, c_1, \dots, c_{n-2} \rangle$ or $G_v = \langle B, b, c_1, \dots, c_{n-2} \rangle$ which permutes the basis elements c_i and c_j . Similarly I_i denotes the automorphism mapping c_i to c_i^{-1} and fixing the other basis elements.

If y and z are elements of a fixed basis of a free group and $\epsilon \in \{\pm 1\}$, then $(y^\epsilon; z)$ is the automorphism fixing all basis elements different from y and sending y to yz if $\epsilon = 1$, and to $z^{-1}y$ if $\epsilon = -1$. Moreover $(y^\pm; z)$ is the automorphism $y \mapsto z^{-1}yz$ fixing the other basis elements.

The following is the special case $k = 2$ of Proposition 7.1 in [10]. Let $y_1 := B = aba^{-1}$ and $y_2 := b$.

Proposition 8.1. *A presentation of $\text{Aut}(G_v, \mathcal{C}_v)$ is given by*

Generators:

- $P_{i,j}$ for $1 \leq i, j \leq n-2$ and $i \neq j$,
- I_i for $1 \leq i \leq n-2$,
- $(c_i^\epsilon; z)$ for $1 \leq i \leq n-2$, $\epsilon = \pm 1$ and $c_i \neq z \in \{c_1, \dots, c_{n-2}, y_1, y_2\}$,
- $(y_i^\pm; z)$ for $i \in \{1, 2\}$ and $y_i \neq z \in \{c_1, \dots, c_{n-2}, y_1, y_2\}$.

Relations: Whenever $z, z_i \in \{c_1, \dots, c_{n-2}, y_1, y_2\}$ and $w, w_i = c_{j_i}^{\delta_i}$ or $y_{j_i}^\pm$,

- Q1 Relations in $\text{Aut}(F_{n-2})$ for $\{(c_i^\epsilon; c_j), P_{i,j}, I_j\}$,
- Q2 $(w_1; z_1)(w_2; z_2) = (w_2; z_2)(w_1; z_1)$ for $w_1 \neq w_2$ and $z_i^{\pm 1} \notin \{w_1, w_2\}$,
- Q3.1 $(y_i^\pm; c_j)P_{j,l} = P_{j,l}(y_i^\pm; c_l)$,
- Q3.2 $(y_i^\pm; c_j)I_j = I_j(y_i^\pm; c_j^{-1})$,
- Q3.3 $P_{j,l}, I_j$ commute with $(y_1^\pm; y_2), (y_2^\pm; y_1)$,
- Q3.4 $(c_j^\epsilon; y_i)P_{j,l} = P_{j,l}(c_j^\epsilon; y_i)$,
- Q3.5 $(c_j; y_i)I_j = I_j(c_j^{-1}; y_i)$,
- Q4.1 $(w; c_j^{-\eta})(c_j^\eta; z)(w; c_j^\eta) = (w; z)(c_j^\eta; z)$,
- Q4.2 $(y_i^\pm; z^{-\epsilon})(w; y_i)(y_i^\pm; z^\epsilon) = (w; z^\epsilon)(w; y_i)(w; z^{-\epsilon})$,
- Q5 $(c_j^{-\eta}; y_i)(y_i^\pm; c_j^\eta) = (y_i^\pm; c_j^\eta)(c_j^\eta; y_i^{-1})$,

whenever the symbols involved denote generators or inverses.

Here we read compositions from right to left. We warn the reader that the articles [10] and [15] use the opposite convention.

In the original article [10] the relations Q3.3 through Q3.5 are missing. It was independently noticed by Andrew Putman and by the first author that these relations have to be added.

8.3. The presentation for $C(\rho)$. By definition, $C^0(\rho)$ is the kernel of the map $\bar{\beta}$ of Proposition 5.3. Since Γ is a loop with a single vertex, $\text{Aut}(\Gamma) = \text{Aut}(\Gamma, v)$ is cyclic of order 2 generated by the isometry

swapping e with \bar{e} . The map $\bar{\beta}$ is surjective: Indeed, for $\theta \in C(\rho)$ defined by

$$\theta(x) = \begin{cases} a^{-1}, & \text{if } x = a, \\ ab^{-1}a^{-1}, & \text{if } x = b, \\ c_i, & \text{if } x = c_i, \end{cases}$$

the graph automorphism $\bar{\beta}(\theta)$ is non-trivial, as a is identified with t_e . Hence $C^0(\rho)$ is a subgroup of index 2 in $C(\rho)$.

For $z \in F_n$ let γ_z be defined by

$$\gamma_z(x) = \begin{cases} az, & \text{if } x = a, \\ z^{-1}bz, & \text{if } x = b, \\ c_i, & \text{if } x = c_i. \end{cases}$$

For notational convenience we sometimes write $(\gamma_\bullet; z)$ instead of γ_z .

Theorem 8.2. *The centraliser of the Nielsen automorphism ρ has the following presentation:*

Generators:

$$\begin{aligned} P_{i,j} & \quad \text{for } 1 \leq i, j \leq n-2 \text{ and } i \neq j, \\ I_i & \quad \text{for } 1 \leq i \leq n, \\ (c_i^\epsilon; z) & \quad \text{for } 1 \leq i \leq n-2, \epsilon = \pm 1, \text{ and } c_i \neq z \in \{aba^{-1}, b, c_1, \dots, c_{n-2}\}, \\ \gamma_z & \quad \text{for } z \in \{aba^{-1}, c_1, \dots, c_{n-2}\}, \\ (a^{-1}; z) & \quad \text{for } z \in \{b, c_1, \dots, c_{n-2}\}, \\ \rho, \\ \theta. \end{aligned}$$

Relations: Given elements $z, z_i \in \{aba^{-1}, b, c_1, \dots, c_{n-2}\}$ and $u, u_i \in \{c_1^{\pm 1}, \dots, c_{n-2}^{\pm 1}, a^{-1}, \gamma_\bullet\}$:

$$\begin{aligned} R1 & \quad \text{Relations in } \text{Aut}(F_{n-2}) \text{ for } \{(c_i^\epsilon; c_j), P_{i,j}, I_j\}, \\ R2 & \quad (u_1; z_1)(u_2; z_2) = (u_2; z_2)(u_1; z_1) \text{ for } u_1 \neq u_2 \text{ and } z_i^{\pm 1} \notin \{u_1, u_2\}, \\ R3.1 & \quad (a^{-1}; c_j)P_{j,l} = P_{j,l}(a^{-1}; c_l), \\ R3.2 & \quad (a^{-1}; c_j)I_j = I_j(a^{-1}; c_j^{-1}), \\ R3.3 & \quad \gamma_{c_j} \circ P_{j,l} = P_{j,l} \circ \gamma_{c_l}, \\ R3.4 & \quad \gamma_{c_j} \circ I_j = I_j \circ \gamma_{c_j}^{-1}, \\ R3.5 & \quad P_{j,l}, I_j \text{ commute with } (a^{-1}; b) \text{ and } \gamma_{aba^{-1}}, \\ R3.6 & \quad (c_i^\epsilon; z)P_{j,l} = P_{j,l}(c_i^\epsilon; z) \text{ for } z = aba^{-1} \text{ or } z = b, \\ R3.7 & \quad (c_j; z)I_j = I_j(c_j^{-1}; z) \text{ for } z = aba^{-1} \text{ or } z = b, \\ R4.1 & \quad (u; c_j^{-\eta})(c_j^\eta; z)(u; c_j^\eta) = (u; z)(c_j^\eta; z), \\ R4.2 & \quad (a^{-1}; z^{-\epsilon})(u; aba^{-1})(a^{-1}; z^\epsilon) = (u; z^\epsilon)(u; aba^{-1})(u; z^{-\epsilon}), \\ R4.3 & \quad \gamma_z^{-\epsilon}(u; b)\gamma_z^\epsilon = (u; z^\epsilon)(u; b)(u; z^{-\epsilon}), \\ R5.1 & \quad (c_j^{-\eta}; aba^{-1})(a^{-1}; c_j^\eta) = (a^{-1}; c_j^\eta)(c_j^\eta; ab^{-1}a^{-1})\rho, \\ R5.2 & \quad (c_j^{-\eta}; b)\gamma_{c_j}^\eta \circ \rho = \gamma_{c_j}^\eta(c_j^\eta; b^{-1}), \end{aligned}$$

R6 ρ commutes with all generators,

R7 $\theta^2 = 1$,

R8.1 $\theta \circ P_{i,j} = P_{i,j} \circ \theta$,

R8.2 $\theta \circ I_i = I_i \circ \theta$,

R8.3 $\theta \circ (c_i^\epsilon; c_j) = (c_i^\epsilon; c_j) \circ \theta$,

R8.4 $\theta \circ (c_i^\epsilon; aba^{-1}) = (c_i^\epsilon; b^{-1}) \circ \theta$,

R8.5 $\theta \circ \gamma_{c_i} = (a^{-1}; c_i) \circ \theta$,

R8.6 $\theta \circ \gamma_{aba^{-1}} = (a^{-1}; b^{-1}) \circ \theta$,

whenever the symbols involved denote generators or inverses.

Proof. Recall the short exact sequence (3) on page 25:

$$1 \rightarrow \langle \rho \rangle \rightarrow C^0(\rho) \rightarrow \text{Aut}(G_v, \mathcal{C}_v) \rightarrow 1.$$

To get a presentation for $C^0(\rho)$, we now use Proposition 7.1. We first have to lift the generators of $\text{Aut}(G_v, \mathcal{C}_v)$ to $C^0(\rho)$. Since the surjection in the short exact sequence is given by restriction to the vertex group G_v , lifting means extending automorphisms from G_v to all of F_n . The generators $P_{i,j}$, I_i , and $(c_i^\epsilon; c_j)$ are lifted to the elements of $C^0(\rho)$ called by the same name. The elements $(c_i^\epsilon; y_1)$ and $(c_i^\epsilon; y_2)$ are lifted to $(c_i^\epsilon; a)$ and $(c_i^\epsilon; aba^{-1})$ respectively. The generator $(y_1^\pm; z)$ can be extended to $(a^{-1}; z)$ and $(y_2^\pm; z)$ to γ_z .

We now have a generating set for $C^0(\rho)$ consisting of these lifted generators together with ρ . The lifted relations are R1 through R5, which have the same numbers as the corresponding relations in Proposition 8.1. A direct calculation shows that ρ only shows up in R5. Since ρ is central, the conjugation relations are simply the commutation rules R6. As the left hand term in this exact sequence is simply the infinite cyclic group generated by ρ , there are no kernel relations.

To get a presentation for $C(\rho)$, we apply Proposition 7.1 to the short exact sequence

$$1 \rightarrow C^0(\rho) \rightarrow C(\rho) \xrightarrow{\bar{\beta}} \mathbb{Z}/2\mathbb{Z} \rightarrow 1,$$

where $\mathbb{Z}/2\mathbb{Z}$ is identified with $\text{Aut}(\Gamma, v)$, so that $\bar{\beta}(\theta) = -1$. A generating set for $C(\rho)$ is then given by θ and our chosen generators of $C^0(\rho)$. The relations are again R1 through R6 along with the lifted relation $\theta^2 = 1$ coming from $\mathbb{Z}/2\mathbb{Z}$, which we label R7, and the conjugation relations given by R8. \square

8.4. The abelianisation. For an element g in an arbitrary group G let $\llbracket g \rrbracket$ denote its class in the abelianisation $H_1(G) = G/[G, G]$. We now study the abelianisation of $C(\rho)$.

Corollary 8.3. *Let $\rho \in \text{Aut}(F_n)$ be a Nielsen automorphism. Then*

$$H_1(C(\rho)) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3, & \text{if } n = 3, \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{if } n = 4, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } n \geq 5. \end{cases}$$

When $n = 2$, the class $[\rho]$ is a generator of \mathbb{Z}^2 , when $n = 3$ it is twice a generator of \mathbb{Z} , and otherwise $[\rho] = 0$.

Proof. We abelianise the presentation in Theorem 8.2. We first restrict to the case $n = 2$: the generators of $C(\rho)$ in this case are $\gamma_{aba^{-1}}$, $(a^{-1}; b)$, ρ and θ . The only relations which occur and are non-trivial in the abelianisation are R7 and R8.6, which become $2[\theta] = 0$ and $[\gamma_{aba^{-1}}] + [(a^{-1}; b)] = 0$. This finishes the proof of the assertion for $n = 2$.

Next we consider $n = 3$. For simplicity we write $c := c_1$. Here the generators of $C(\rho)$ are $I := I_1$, $(c^\epsilon; aba^{-1})$, $(c^\epsilon; b)$, $\gamma_{aba^{-1}}$, γ_c , $(a^{-1}; b)$, $(a^{-1}; c)$, ρ and θ . From Theorem 8.2 we obtain the following relations:

$$\begin{array}{ll} \text{R1:} & 2[I] = 0 \\ \text{R3.2:} & 2[(a^{-1}; c)] = 0 \\ \text{R3.4:} & 2[\gamma_c] = 0 \\ \text{R3.7:} & [(c; aba^{-1})] = [(c^{-1}; aba^{-1})] \\ \text{R3.7:} & [(c; b)] = [(c^{-1}; b)] \\ \text{R4.1} & [\gamma_{aba^{-1}}] = 0 \end{array} \quad \begin{array}{ll} \text{R4.1:} & [(a^{-1}; b)] = 0 \\ \text{R5.1:} & [\rho] = 2[(c; aba^{-1})] \\ \text{R5.2:} & -[\rho] = 2[(c; b)] \\ \text{R7:} & 2[\theta] = 0 \\ \text{R8.4:} & [(c; b)] = -[(c; aba^{-1})] \\ \text{R8.5:} & [\gamma_c] = [(a^{-1}; c)] \end{array}$$

All other relations in $H_1(C(\rho))$ either follow from the ones above or are trivial. It follows that $H_1(C(\rho)) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$ with the torsion part generated by $[I]$, $[\gamma_c]$ and $[\theta]$ and the torsion-free part generated by $[(c; b)]$ with $[\rho] = -2[(c; b)]$.

For $n \geq 4$, by checking the relations R1–R8 one finds that there is a homomorphism

$$C(\rho) \rightarrow H_1(\text{Aut}(F_{n-2}))$$

given by sending the elements $(c_i^\epsilon; aba^{-1})$, $(c_i^\epsilon; b)$, γ_z , $(a^{-1}; z)$, ρ and θ to 0 and letting the remaining generators of $C(\rho)$ act on $F_{n-2} = \langle c_1, \dots, c_{n-2} \rangle$. We also have the homomorphism

$$C(\rho) \rightarrow C(\rho)/C^0(\rho) \cong \mathbb{Z}/2\mathbb{Z}$$

that takes every generator except θ to 0. Combining these gives a surjective homomorphism

$$f : C(\rho) \rightarrow H_1(\text{Aut}(F_{n-2})) \oplus C(\rho)/C^0(\rho).$$

The relation R4.1 implies that $\llbracket(u; z)\rrbracket = 0$ if there is a symbol c_j different from both u, z and their inverses. Hence any generator $(u; z)$ not of the form (c_i^ϵ, c_j) is trivial in $H_1(C(\rho))$. Furthermore, as $\llbracket(c_i; aba^{-1})\rrbracket = 0$ we have $\llbracket\rho\rrbracket = 0$ by R5.1. It follows that any non-trivial element of $H_1(C(\rho))$ is nontrivial under the map f , and as imf is abelian, we have $imf \cong H_1(C(\rho))$. We finish the proof with the observation that:

$$H_1(\text{Aut}(F_{n-2})) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } n = 4, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } n \geq 5. \end{cases}$$

This may be found by abelianising one's favourite presentation of $\text{Aut}(F_n)$ (from [14], [16], or [17], say). \square

Remark 8.4. Given a basis a_1, \dots, a_n of F_n and $w \in \langle a_2, \dots, a_n \rangle$, we can define $\rho \in \text{Aut}(F_n)$ by $a_1 \rightarrow a_1 w$ and fixing a_2, \dots, a_n . Abelianisations of centralisers of these more general right translations are computed in [18].

8.5. Connection to CAT(0) actions. A *CAT(0) space* is a geodesic metric space (X, d) whose geodesic triangles are not thicker than euclidean comparison triangles with the same sidelengths (cf. [4] for a more precise definition). The *translation length* of an isometry $\gamma : X \rightarrow X$ is defined by

$$|\gamma| = \inf_{x \in X} d(x, \gamma(x)).$$

The work above relates to the following theorem, which appears implicitly in the proof of Theorem 2.6 in [2]:

Theorem 8.5 (Bridson, Karlsson, Margulis). *Let G be any group and $g \in G$. Assume that $\llbracket g \rrbracket$ has finite order in $H_1(C(g))$. Then $|g| = 0$ whenever G acts by isometries on a proper CAT(0) space.*

In Corollary 8.3 we have seen that $\llbracket \rho \rrbracket$ has infinite order in $H_1(C(\rho))$ if $n \leq 3$. In fact, in Section 6 of [3] there is a construction for isometric CAT(0) actions of $\text{Aut}(F_3)$ such that Nielsen automorphisms act by positive translation length. However, for $n \geq 4$ we have seen that $\llbracket \rho \rrbracket = 0$. Hence:

Corollary 8.6. *If $n \geq 4$, Nielsen automorphisms always act by zero translation length whenever $\text{Aut}(F_n)$ acts isometrically on a proper CAT(0) space.*

This sharpens a result of Bridson [3], who proved Corollary 8.6 when $n \geq 6$ by showing that $\llbracket \rho \rrbracket = 0$ when $n \geq 6$ using the lantern relation in mapping class groups, rather than a direct computation of $H_1(C(\rho))$.

The first author has a short geometric proof of Corollary 8.6, via a parallelogram formula

$$|\alpha\beta|^2 + |\alpha\beta^{-1}|^2 = 2(|\alpha|^2 + |\beta|^2)$$

for the translation lengths of two commuting isometries α and β (proved in [18]). If we define α and β like so:

$$\begin{array}{ll} \alpha: a \mapsto a, & \beta: a \mapsto ab, \\ b \mapsto b, & b \mapsto b, \\ c \mapsto cb, & c \mapsto c, \\ d \mapsto db^{-1}, & d \mapsto d, \end{array}$$

then α is then conjugate to both $\alpha\beta$ and $\alpha\beta^{-1}$. Hence these three automorphisms have the same translation length. The parallelogram formula then implies that $|\beta| = 0$.

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Moritz Rodenhausen
Mathematisches Institut
Endenicher Allee 60
53115 Bonn
Germany
rodenhau@math.uni-bonn.de

Richard D. Wade
Mathematical Institute
24-29 St Giles'
Oxford OX1 3LB
United Kingdom
wade@maths.ox.ac.uk